The following is a contributed topic on functorial algebraic geometry and physics:

"Functorial Algebraic Geometry: An Introduction" -by Alexander Grothendieck

0.0.1 Vol.1: Affine Algebraic Geometry

(Following the Notes typewritten in English and edited by P. Gaeta, without implying the approval by A. Grothendieck of these notes.)

A century ago algebraic Geometry could be contained in Klein's book (1880; Dover publs. 1963) : "On Riemann's theory of Algebraic Functions and Their Integrals". Furthermore, "*the distinction between pure and applied mathematics was then to a large extent artificial and unimportant*" (viz. P. Gaeta). For example in Klein's book cited above "the study of Riemann surfaces was introduced by considering the practical physical problem of laminar flow in a plane or arbitrary surface. He even quotes Maxwell's treatise on page one. The natural continuation of such a 'transcedental approach' in our times is the study of *complex algebraic manifolds...*" In contrast with Algebraic Geometry, the popular beliefs regarding Differential Geometry are totally different: the latter never lost its flavor of applicability; such practical examples of differentiable manifolds are natural examples of *locally ringed spaces*. "Thus, if a reader is familiar with differentiable manifolds, Grothendieck's *schemes* cannot look so terribly abstract...; we do not assume knowledge of differentiable manifolds as a logical pre-requisite for this course, but a student interested in applications should be interested in differentiable manifolds. The purpose of this informal Introduction is to develop an analogy between these new mathematical objects introduced by Grothendieck (that is, in Algebraic Geometry) and certain objects within the structure of Mathematical Physics..." Consider the 'configuration space' V_n or the 'phase space' W_{2n} of a holonomic dynamical system 'with n-degrees of freedom'; for any problems concerning V_n one should only consider local functions $f : U \to R$ defined within an open set $U \subset V_n$. As an example, a Lagrangian coordinate function q_i (with $i = 1, 2, ..., n$) is only defined locally for a certain coordinate chart. The Lagrange equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{0.1}$$

are valid only in a certain local coordinate system $(q_1, ..., q_n)$. In order to examine the behavior of the dynamic system globally one must piece together local functions corresponding to different sets U, and this is achieved by verifying first that the set of functions $[f : U \to \mathbb{R} | U \subset V_n]$ form a *commutative ring with unit* $\Gamma(U)$ under pointwise addition and multiplication for such U. If $V \subset U$, then there is a natural *restriction map* $r^U_V : \Gamma(U) \to \Gamma(V)$ which assigns to every $\phi : U \to \mathbb{R}$ its restriction map with respect to V, that is, $r_V^U = \phi|V : V \to \mathbb{R}$. This also means–in other words–that the local C^∞- differentiable functions on U form , or define, a '*presheaf*' (viz. Ch.III).

Next one must consider the "*germ*" of $f : U \to \mathbb{R}$ at any point $x \in U$. Thus, let $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ be two such local functions; one then notes that f and g are equivalent functions, $f \cong g$, if they agree on $W \subset U \cap V | x \, not \, in \subset U \cap V$. The germ of f at the point $x \in W$ denoted by $\widetilde{x}$ is the equivalence class of functions determined by this $\cong$ relation. One notes that this definition appears in elementary 'complex analysis' in one variable. One can readily check that the germs $\widetilde{x}$ for all $x \in W$ form a *local ring* (in the modern sense of the concept). Henceforth, with the addition of several topological properties, one can define a '*sheaf of germs of local* C^∞*-differentiable functions of* M' denoted by Θ_M. Θ_X in the case when X is a topological space can be then defined as the disjoint sum $\cup_{x \in M} \Theta_{M,x}$ of the local rings $\Theta_{M,x}$ for every point of X. Therefore, the differentiable manifold V_n or W_{2n} of classical mechanics (or indeed, any differential manifold) is an example of a locally ringed space (X, Θ_x), that is a topological space X with a structure sheaf Θ_X. Grothendieck's *schemes* are also **locally ringed spaces** (X, Θ_X).

Thus, sheaves were introduced to provide a transition from *local to global* properties. Therefore, "the global study of curves which solve the classical equations of motion–which is a difficult problem–has been simplified by the introduction of sheaves".

Following Dieudonné 's and Grothendieck's famous "Élements de Géometrie Algébrique", and Dieudonné 's "Algebraic Geometry" and "Fondements de la Géometrie Algébrique." Adv. in Math. (1969), Alexander Grothendieck presented in 1973 a Buffalo Summer Course entitled: "Survey on the functorial approach to affine algebraic groups". This was preceded by a lecture introducing the functorial 'language' approach (Introduction au Langage Fonctoriel)

Grothendieck also organized and presented most of the four famous SGA seminars (SGA-1 to SGA-4), "Séminaires de Géometrie Algébrique" (Seminars of Algebraic Geometry.) . Other relevant references were: Kähler's "Geometria arithmetica" (1958), S. MacLane's "Homology" (1963), Manin's "Lectures on Algebraic Geometry", Mumford's "Introduction to Algebraic Geometry", and J.P. Serre's "Faisceaux algébrique cohérents." (Coherent Algebraic Sheaves). In 1968 was also published by North-Holland the book "Dix exposés sur la cohomologie des schémes" (Ten expositions on the cohomology of schemes) by J. Giraud and Alexander Grothendieck.